Rotation sets and unbounded behavior for toral homeomorphisms

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Introduction

Discrete dynamical systems

- X topological space (usually a compact manifold)
- $f : X \to X$ homeomorphism
- study the orbit structure of the $\mathbb{Z}$-action $\{f^n\}_{n\in\mathbb{Z}}$ (where $f^n = f \circ f \circ \cdots \circ f$).
- $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ the *orbit of* x
- Behavior of $f^n(x)$ as $n \to \infty$ or $-\infty$ (asymptotic behavior of orbits).
- Periodic orbits? Invariant measures? Etc.

Dynamics of one-dimensional homeomorphisms

- $f\colon \mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{T}^1$ orientation-preserving homeomorphism
- Model dynamics: rigid rotation $R_\alpha(x) = x + \alpha \pmod{\mathbb{Z}}$.
  - $\alpha = p/q \pmod{\mathbb{Z}}$ rational $\implies$ all orbits are periodic of period q
  - α irrational $\implies$ all orbits are dense (*minimal* dynamics).
- Poincaré's idea: measure the "average asymptotic rotation" of a general homeomorphism:

$$\rho(f) = \lim_{n\to\infty} \frac{\widehat{f}^n(x) - x}{n} \pmod{\mathbb{Z}}$$

  where $\widehat{f}\colon \mathbb{R} \to \mathbb{R}$ is a lift of f to the universal covering (i.e. $\pi\widehat{f} = f\pi$ where $\pi\colon \mathbb{R} \to \mathbb{T}^1$ is the projection).
- This "rotation number" does not depend on the choices of x or the lift.

Dynamics of one-dimensional homeomorphisms

Theorem (Poincaré)

- $\rho(f) = p/q \pmod{\mathbb{Z}} \implies$ there is a periodic orbit, and all periodic orbits have the same period q
- $\rho(f)$ irrational $\implies$ f is *monotonically semiconjugate* to a rigid rotation, and all orbits have the same limit (which is either a unique cantor set Λ or the whole circle).

"Theorem" (Poincaré)

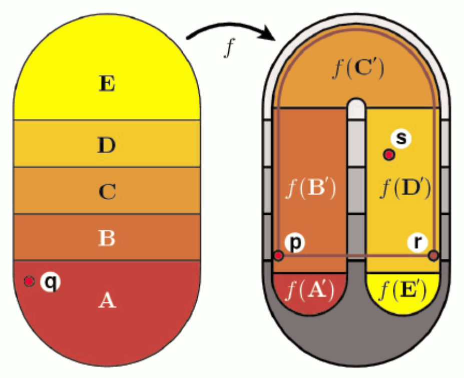
The dynamics of homeomorphisms of $\mathbb{T}^1$ can be completely classified.

Key aspects

- Possible dynamics are *simple*.
- All orbits behave in a relatively similar way.
- Bounded deviations.

Dimension two: explosion of new phenomena.

Example: Smale's horseshoe

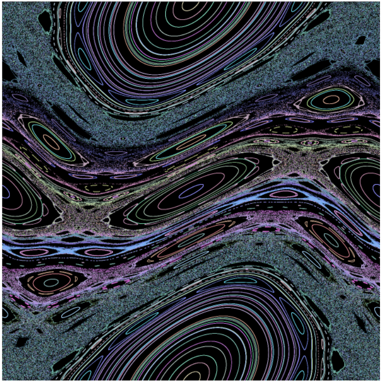


Shows up frequently. Infinitely many periodic orbits (of all periods). Positive entropy. Sensitive dependence on initial conditions; "chaos".

Area-preserving homeomorphisms

For the rest of the talk we will consider area-preserving surface homeomorphisms: $f\colon S \to S$ such that $\mu(f(E)) = \mu(E)$ for all Borel sets E, where μ is the area measure on S.

"Typical" phase portrait:



Rotation in dimension two

Trivial example

$f\colon \mathbb{A} = \mathbb{T}^1 \times [0,1] \to \mathbb{A}, \quad f(x,y) = (x + \sin(2\pi y), y)$
has orbits with many different "average rotation" speeds, and periodic orbits with all kinds of periods.

Poincaré-Birkhoff Theorem

If $f\colon \mathbb{A} \to \mathbb{A}$ preserves area, orientation and boundary components and has rotation numbers of opposite signs in the two boundary circles, then there are fixed points in $\mathbb{A}$.

Corollary

There are inifnitely many periodic points in $\mathbb{A}$ of arbitrarily large periods.

Remark (Birkhoff, Mather)

If there are no essential invariant "curves" then: rich dynamics.

Rotation in dimension two

- $f\colon \mathbb{T}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ homeomorphism *homotopic to the identity*
- Two directions of rotation.
- As in the circle, consider a lift $\widehat{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ to the universal covering, i.e. $\pi\widehat{f} = f\pi$ where $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ is the projection.

The **rotation vector** of z is

$$\rho(\widehat{f}, z) = \lim_{n\to\infty} \frac{\widehat{f}^n(z) - z}{n}.$$

It measures the asymptotic average rotation of $\pi(z)$ along the two homological directions of $\mathbb{T}^2$.

- The limit doesn't always converge;
- When it does, it usually depends on z.

Rotation set (Misiurewicz-Ziemian, 89)

The **rotation set** $\rho(\widehat{f})$ is the set of all limits of the form

$$\lim_{k\to\infty} \frac{\widehat{f}^{n_k}(z_k) - z_k}{n_k}, \qquad \text{with } z_k \in \mathbb{R}^2 \text{ and } n_k \to \infty.$$

The rotation vector of an invariant measure $\mu \in \mathcal{M}(f)$ is

$$\rho_\mu(\widehat{f}) = \int \phi \, d\mu,$$

where ϕ *displacement function* (induced on $\mathbb{T}^2$ by $\widehat{f} - \mathrm{Id}$).

- $\rho(\widehat{f}, z)$ exists μ-a.e z.
- $\rho_\mu(\widehat{f}) = \int \rho(\widehat{f}, z) \, d\mu$.
- $\rho(\widehat{f}) = \{\rho_\mu(\widehat{f}) : \mu \in \mathcal{M}(f)\}$.
- It is compact and convex, and it is the convex hull of the set of rotation vectors of points.

Shape of the rotation set

- Which compact convex sets are rotation sets?
  - Single point sets;
  - Some intervals;
  - Convex polygons with rational vertices (Kwapisz, 1995)
  - There is an example which is not a polygon (but almost);
  - That's about all that is known.
- Is there **some** compact convex set which is not a rotation set? Recent result (Tal, Le Calvez 2016): Yes, a specific interval.
- Is there some compact convex set **nonempty interior** which is not a rotation set?
- Can the rotation set have uncountably many extremal points?
- Can it be a circle?

Sublinear rotation

$\rho(\widehat{f}, z) = v \implies$ the orbit of z escapes towards ∞ with average velocity v. If the orbit of z escapes *sublinearly* (e.g. if $|\widehat{f}^n(z) - z| = (\sqrt{n}, 0)$) then $\rho(\widehat{f}, z) = (0, 0)$ (i.e. the rotation vector does not distinguish it from a fixed point).

General idea

Rotation vectors in two opposite directions $\implies$ there is no sublinear rotation in the transverse direction.

Examples: Mather '91 and Slijepcevic '01 (twist maps), Bortolatto and Tal '12 (certain ergodic maps), Addas-Zanata, Garcia and Tal '13 (Dehn isotopy class).

Sublinear rotation

Theorem (Guelman, K., Tal '13)

If $\rho(\widehat{f}) \subset \{0\} \times \mathbb{R}$ and it has more than one point, then there is no horizontal rotation at all. Specifically, there is an invariant vertical annulus, and so $\sup_{z\in\mathbb{R}^2, n\in\mathbb{Z}} |(\widehat{f}^n(z) - z)_1| < \infty$ (i.e. uniformly bounded horizontal displacement).

In other words, there is a dichotomy:

- Either the dynamics is reduced to an annular dynamics, or
- there are three points with non-collinear rotation vectors

The latter case means the dynamics is extremely rich (see later).

Remark

This was (mostly) generalized removing the area preservation, by P. Dávalos, with a completely different proof.

Sublinear rotation: irrotational homeomorphisms

In the circle, null rotation number $\implies$ uniformly bounded displacement. A similar property does not hold on $\mathbb{T}^2$.

Example with sublinear diffusion (K., Tal '12)

There is a C^∞ ergodic diffeomorphism of $\mathbb{T}^2$ such that $\rho(\widehat{f}) = \{(0,0)\}$ but almost every orbit accumulates on all directions at infinity.

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Objection: the example has a **very** large set of fixed points (a fully essential continuum, i.e. the complement of a disjoint union of open topological disks) which is topologically "bad" (e.g. not locally connected). This is the only possibility.

Theorem (K., Tal '13)

In order to have such an example the set of fixed points *must* be *large and nasty*. More precisely: if $\rho(\widehat{f}) = \{(0,0)\}$ then either f is "annular" or $\mathrm{Fix}(f)$ is fully essential and non-locally connected.

Recently improved by Tal and Le Calvez (2016).

Fat rotation sets

If $\rho(\widehat{f})$ has nonempty interior, then:

- Positive topological entropy (Llibre-McKay '91)
- Abundance of periodic orbits and invariant sets:
  - Every extremal or interior element of $\rho(\widehat{f})$ with rational coordinates is realized by a periodic point (Franks '89)
  - For every $v \in \mathrm{int}(\rho(\widehat{f}))$ there is a compact invariant set K_v with rotation vector v. (Misiurewicz-Zieman '91)
  - more!

Remark

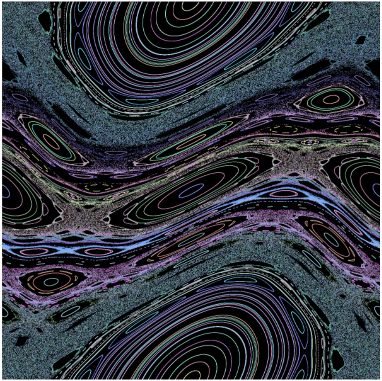
- $\rho(\widehat{f})$ has nonempty interior $\iff$ there are three points with non-collinear rotation vectors;
- "strictly toral" dynamics;
- "typical" for area-preserving maps (C^r-generic, any $r \geq 0$).

Motivation

Typical figure in area-preserving dynamics: many "elliptic islands" and a complementary "instability region" with rich dynamics.

Chirikov-Taylor Standard Map

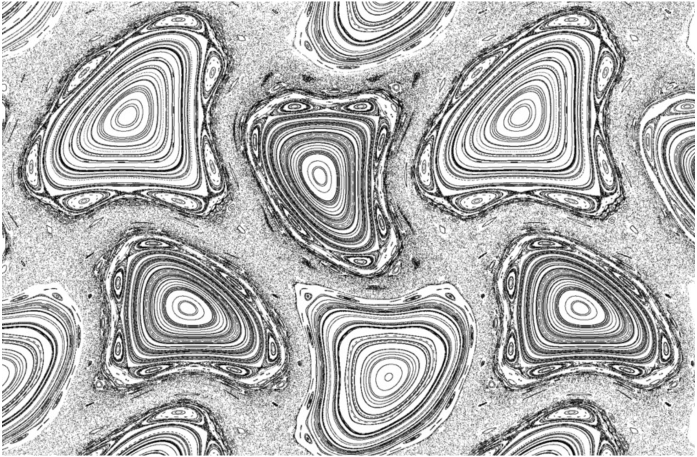
$f : \mathbb{T}^2 \to \mathbb{T}^2$ induced by $\widehat{f}(x, y) = (x + y, y + \alpha \sin(2\pi(x)))$.



Motivation

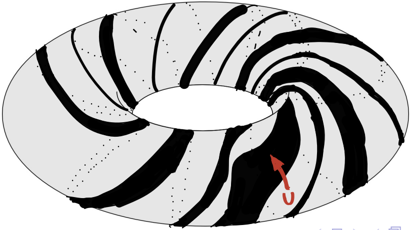
Zaslavsky Web Map

$M\colon \mathbb{T}^2 \to \mathbb{T}^2$ lifted by $\widehat{M}(x,y) = (y, -x - \alpha \sin(2\pi y - \beta))$
and $f = M^4$ (btw: rotation set has nonempty interior)



Motivation

- Kolmogorov-Arnold-Moser (KAM) theory provides a local explanation for the existence of elliptic islands under certain condition for regular maps.
- For instance: for a C^r-generic diffeomorphism (r large), any elliptic fixed point is the intersection of a nested sequence (D_i) of invariant topological disks bounded by circles with irrational rotation numbers. Each D_i contains hyperbolic periodic points with homoclinic intersections, etc. [Moser, Zehnder]
- Global picture? How rich is the dynamics outisde elliptic islands?
- Can we define "maximal" islands? Are they bounded?



"Theorems"

Periodic island = periodic open topological disk U.
We say that U is (homotopically) **bounded** if
$\mathcal{D}(U) = \{$diameter of a lift of U to the universal covering$\} < \infty$

"Theorem"

The general picture of a partition of the space into **bounded** periodic islands and a "large" complementary region with interesting dynamics holds whenever f is "strictly" toral.

Particular case

Homeomorphisms of $\mathbb{T}^2$ with a rotation set with interior. Generic.

"Theorem"

In general, in order to have an unbounded island, the fixed point set must be large (*essential*: not deformable to a point).

Precise statement

Theorem (K., Tal '13)

If $\mathrm{int}(\rho(\widehat{f})) \neq \emptyset$ and f is area-preserving then there exists a partition of $\mathbb{T}^2$ into two sets, $\mathcal{C}(f)$ and $\mathcal{I}(f)$, where:

- $\mathcal{I}(f)$ is a disjoint union of periodic bounded open topological disks ("periodic islands"). Consists of all points which belong to some periodic island.
- $\mathcal{C}(f)$ is connected, weakly transitive, has sensitive dependence on initial conditions, positive entropy ("chaotic region");

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- $\rho(\widehat{f}, B_\epsilon(z)) = \rho(\widehat{f})$ for all $z \in \mathcal{C}(f)$ ("uniform diffusion");
- Every rotation vector realized by a periodic point [ergodic measure, compact invariant set] is also realized by a periodic point [ergodic measure, compact invariant set] in $\mathcal{C}(f)$ ("rotational dynamics is realized in $\mathcal{C}(f)$").

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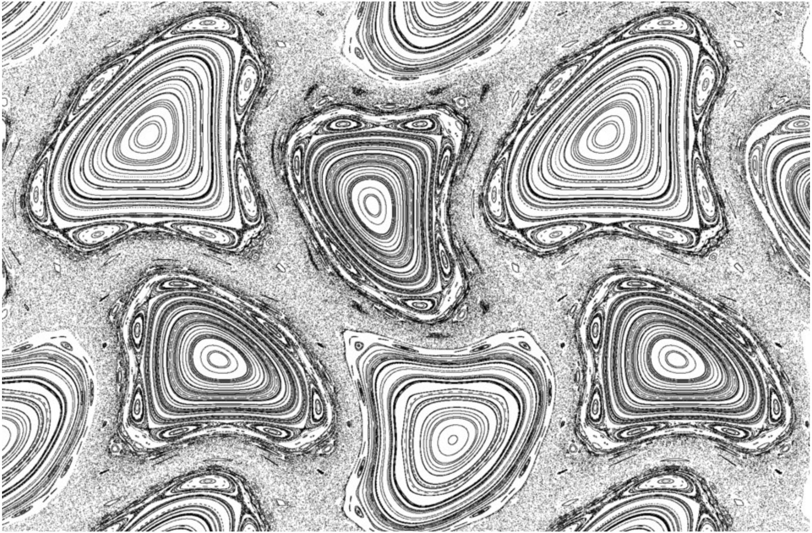
$\mathcal{C}(f)$ was already studied by Jäger (different definition). Key obstruction to conclude many properties: unbounded islands. Addas-Zanata '13: if f is $C^{1+\alpha}$, the bound is uniform.

Example



Transitivity

Corollary

If f is transitive (i.e. has a dense orbit) and $\rho(\widehat{f})$ has interior, there are no islands at all.

Theorem (Tal '12; Guelman, K., Tal '13)

f transitive and $(0,0) \in \mathrm{int}(\rho(\widehat{f})) \iff \widehat{f}$ transitive.

Key Result: “Bounded Disks Lemma”

Bounded disks lemma (K., Tal ’13)

Let f be a nonwandering homeomorphism homotopic to the identity such that $\mathrm{Fix}(f)$ *is inessential.* Then all f-invariant open topological disks are (uniformly) bounded.

[True on any surface, and on any homotopy class in $\mathbb{T}^2$]

Remark

There exists a C^∞ area-preserving ergodic diffeomorphism of $\mathbb{T}^2$ homotopic to Id with an invariant island U such that any lift of U of $\mathbb{R}^2$ intersects every fundamental domain.

Further results and problems

- Similar results for abelian actions (Benayon PhD thesis, 2013)
- Surfaces of higher genus (K.-Tal, 2015)
- The "bounded disks lemma" leads to the following "triple boundary lemma"
  any point in the boundary of three pairwise disjoint invariant connected open sets on the sphere must be a fixed point.
  The latter has many consequences. For example: if the "lakes of Wada" continuum is invariant by an area-preserving map f, then it is fixed pointwise by f^3.
  (work in progress with Tal and Le Calvez).

Problems

- Uniform boundedness of islands, independent of periods? (Addas-Zanata: True for $C^{1+\alpha}$)
- Irrotational homeomorphisms: is sublinear diffusion possible in arbitrary surfaces?

References

- N. Guelman, A. Koropecki and F. A. Tal, *Rotation sets with nonempty interior and transitivity in the universal covering*, Ergodic Theory Dynam. Systems, 2013.
- N. Guelman, A. Koropecki and F. A. Tal, *A characterization of annularity for area-preserving toral homeomorphisms*, Math. Zeitchrift, 2013.
- A. Koropecki and F. A. Tal, *Strictly toral dynamics*, Invent. Math., 2013.
- A. Koropecki and F. A. Tal, *Area-preserving irrotational diffeomorphisms of the torus with sublinear diffusion*, to appear in Proc. Amer. Math. Soc.
- A. Koropecki and F. A. Tal, *Bounded and unbounded behavior for area-preserving rational pseudo-rotations*, preprint, 2012.
- F. A. Tal *Transitivity and rotation sets with nonempty interior for homeomorphisms of the 2-torus*. Proc. Amer. Math. Soc., 2012.